

## Lecture 21 (April 18, 2016)

Stability of perturbed system  $\dot{x} = f(t, x) + g(t, x)$

Assume both  $f$  &  $g$  are piecewise continuous in  $t$  and locally Lipschitz in  $x$  on  $[0, \infty] \times D$  &  $D \subseteq \mathbb{R}^n$ .

$g(t, x)$ : Perturbation term models errors, aging, uncertainties, disturbances.

Don't know  $g(t, x)$ , but will assume we know something about it, e.g. upper bound of  $\|g(t, x)\|$ .

uncertainties that don't change order of system can always be represented by additive term:

$$\tilde{f}(t, x) = f(t, x) + \underbrace{[\tilde{f}(t, x) - f(t, x)]}_{g(t, x)}$$

Suppose  $x=0$  is u.a.s or u.e.s eq. pt of  $\dot{x} = f(t, x)$ .

What can we say about stability of perturbed system?

Natural approach: Use a Lyapunov function for the nominal system as a Lyapunov function candidate for the perturbed system.

(We used this technique when we studied linearization.)

Example. Interconnected Systems.

$$\dot{x}_i = f_i(t, x_i) + g_i(t, x) \quad i=1, \dots, m,$$

$$x_i \in \mathbb{R}^{n_i}, n_1 + \dots + n_m = n, x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^n$$

$$\text{Suppose } f_i(t, 0) = g_i(t, 0) = 0 \quad \forall i.$$

Unperturbed system is  $\dot{x}_i = f_i(t, x_i)$ .

Suppose origin is u.a.s. for each isolated subsystem.

What is stability of interconnected system?

Vanishing ( $g(t,0)=0$ ) and nonvanishing ( $g(t,0)\neq 0$ ) perturbations

Vanishing perturbation  $\dot{x} = f(t,x) + g(t,x)$

$f(t,0)=0$ . Assume:

- $g(t,0)=0$  linear growth bound
- $\|g(t,x)\| \leq \gamma \|x\|, \forall t \geq 0, \forall x \in D$ , for some  $\gamma > 0$ . (\*)
- $x=0$  is an exp. stable eq. pt of  $\dot{x}=f(t,x)$ .

Note that any  $g(t,x)$  that vanishes at the origin, and is locally Lipschitz in  $x$ , uniformly int,  $\forall t \geq 0$ , in a bounded neighborhood.

satisfies  $\|g(t,x) - g(t,0)\| \leq \gamma \|x\|$  in that neighborhood.

However, for (\*) to be satisfied on  $D$ ,  $g$  must be globally Lipschitz.

Let  $V(t,x)$  be Lyapunov function for nominal system that satisfies

$$c_1 \|x\|^2 \leq V(t,x) \leq c_2 \|x\|^2 \quad (1)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3 \|x\|^2 \quad (2)$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\| \quad (3)$$

for all  $(t,x) \in [0,\infty) \times D$  and for some positive constants  $c_i$ 's.

(Existence of such  $V$  is guaranteed by Thm 4.14)

① Robustness of exponential stability of origin

Lemma 9.1. Suppose  $\gamma < \frac{c_3}{c_4}$ . Then the origin is e.s eq.pt of the perturbed system. (globally if assumptions hold globally).

Proof. Use Lyapunov function for nominal system ( $V$ ) on Perturbed sys.

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x} g(t,x) \leq -c_3 \|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \|g(t,x)\| \\ &\leq -c_3 \|x\|^2 + c_4 \|x\| \cdot \gamma \|x\| = -(c_3 - c_4 \gamma) \|x\|^2 \quad c_3 - c_4 \gamma > 0 \end{aligned}$$

$$\Rightarrow \dot{V} \leq - (c_3 - c_4 \gamma) \|x\|^2 \quad (4)$$

By Thm 4.10, (1) & (4)  $\Rightarrow$  0: e.s.

This shows that exponential stability of origin is robust w.r.t. certain perturbations. Don't have to know  $V(t,x)$ . Although in this case don't know  $c_3/c_4$ .

*Example 9.1.*

$$\dot{x} = Ax + g(t,x), \quad A: \text{Hurwitz}, \quad \|g(t,x)\| \leq \gamma \|x\| \quad t \geq 0, \quad x \in \mathbb{R}^n$$

$x=0$  is g.e.s. for  $\dot{x}=Ax$ . Want to show that it is g.e.s. for  $\dot{x}=Ax+g(t,x)$ .

Let  $Q=Q^T > 0$  and solve LE  $PA + A^T P = -Q$  for  $P$ .

Thm 4.6 guarantees a unique  $P$ .

Let  $V = x^T Px$ . Then  $V$  satisfies (1) - (3) with

$$c_1 = \lambda_{\min}(P), \quad c_2 = \lambda_{\max}(P)$$

$$c_3 = \lambda_{\min}(Q) : \frac{\partial V}{\partial x} \cdot Ax = -x^T Q X \leq -\lambda_{\min}(Q) \|x\|^2$$

$$c_4 = 2\lambda_{\max}(P) \quad \left\| \frac{\partial V}{\partial x} \right\| = \|2x^T P\| \leq 2\|P\| \|x\| \leq 2\lambda_{\max}(P) \|x\|$$

For perturbed system:

$$\dot{V} = -x^T Q X + 2x^T P g(t,x) \leq -\lambda_{\min}(Q) \|x\|^2 + 2\lambda_{\max}(P) \gamma \|x\|^2$$

So  $x=0$  is g.e.s. if  $\gamma < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$ .

How to choose  $Q$  to maximize  $\gamma$ ?  $Q=I$ . (Exe 9.1.)

*Example 9.2.*  $\dot{x}_1 = x_2$

$$\dot{x}_2 = -4x_1 - 2x_2 + \beta x_2^3, \quad \beta \geq 0 \quad \text{is unknown}$$

i.e.  $\dot{x} = Ax + g(x)$ , where  $A = \begin{pmatrix} 0 & 1 \\ -4 & -2 \end{pmatrix}$  &  $g(x) = \begin{pmatrix} 0 \\ \beta x_2^3 \end{pmatrix}$

$$\text{eig}(A) = -1 \pm \sqrt{3}i \Rightarrow A: \text{Hurwitz}.$$

How big can we make  $\beta$  and still remain exp. stability of  $x=0$  (at least locally) ?

Let  $Q=I$ . Then  $P = \begin{pmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{pmatrix}$  (Matlab: lyap)

$$\|g(x)\| = \beta |x_2|^3 \leq \beta K_2^2 |x_2| \leq \beta K_2^2 \|x\|^2 \quad \forall |x_2| < K_2 \iff \delta = \beta K_2^2$$

(we can't yet know  $K_2$ , but certainly  $|x_2|$  is bounded if  $x(t)$  is confined to a compact set)

$\dot{v}$  for perturbed system:

$$\begin{aligned} \dot{v} &\leq -\lambda_{\min}(Q) \|x\|^2 + 2\lambda_{\max}(P) v \|x\|^2 \\ &= -\|x\|^2 + 3.026 \beta K_2^2 \|x\|^2 < 0 \quad x \neq 0 \end{aligned}$$

$$\text{for } -1 + 3.026 \beta K_2^2 < 0 \iff \beta < \frac{1}{3.026 K_2^2}$$

Now let's estimate  $K_2$ :

$$\Omega_c = \left\{ x \in \mathbb{R}^2 \mid V(x) = \frac{3}{2}x_1^2 + \frac{1}{4}x_1x_2 + \frac{5}{16}x_2^2 \leq c \right\}$$

Consider the surface  $V(x) = \frac{3}{2}x_1^2 + \frac{1}{4}x_1x_2 + \frac{5}{16}x_2^2 = c$ . Want to find  $\max |x_2|$  for given  $c$ .  $x_2$  depends on  $x_1$ , so set  $\frac{\partial x_2}{\partial x_1} = 0$  (gives critical points) and solve for  $x_1$  in terms of  $x_2$ .

$$\frac{\partial x_2}{\partial x_1} = \frac{\partial x_2}{\partial V} \cdot \frac{\partial V}{\partial x_1} = 0 \quad \text{when} \quad \frac{\partial V}{\partial x_1} = 0.$$

$$\frac{\partial V}{\partial x_1} = 3x_1 + \frac{1}{4}x_2 = 0 \iff x_1 = -\frac{1}{12}x_2$$

Look at intersection of this with  $V(x) = c$ :

$$\frac{3}{2}\left(\frac{1}{144}x_2^2\right) - \frac{1}{48}x_2^2 + \frac{5}{16}x_2^2 = c \iff \frac{29}{96}x_2^2 = c \iff |x_2| = \sqrt{\frac{96c}{29}}$$

Thus if  $x \in \Omega_c$ , then  $|x_2| < K_2$ ,  $K_2 = \sqrt{\frac{96c}{29}}$ .

So if  $\beta < \frac{1}{3.026} \frac{29}{96c} \approx \frac{0.1}{c}$   $\Rightarrow \dot{v} < 0$  on  $\Omega_c$ ,  $x \neq 0$  and  $x=0$  is e.s.

$\Omega_c$  is estimate of region of attraction.

Note tradeoff between  $\beta$  &  $c$ . Smaller  $c$  (region of attraction), larger perturbation allowed.

We can get less conservative bound if we explicitly consider structure of perturbation: (i.e. don't get in terms of  $\gamma$  but just apply it directly to perturbed system.)

$$\begin{aligned}\dot{V} &= -\|x\|^2 + 2x^T P g(x) \\ &= -\|x\|^2 + 2\beta x_2^2 \left( \frac{1}{8} x_1 x_2 + \frac{5}{16} x_2^2 \right) \\ &\leq -\|x\|^2 + 2\beta x_2^2 \left( \frac{1}{16} \|x\|^2 + \frac{5}{16} \|x\|^2 \right) \quad \left( 0 < (x_1 - x_2)^2 = x_1^2 + x_2^2 - 2x_1 x_2 \right) \\ &\leq -\|x\|^2 + \frac{3}{4} \beta K_2^2 \|x\|^2\end{aligned}$$

$$\text{so } \dot{V} < 0 \text{ for } \beta < \frac{4}{3K_2^2}$$

$$\text{so for } x \in \Omega_c, |x_2| \leq K_2 = \sqrt{\frac{96c}{29}} \quad \dot{V} < 0 \text{ if } \beta < \frac{0.4}{c} \quad (\text{4 times previous})$$

② difficult situation: robustness of u.a.s-ity of the origin.

$x=0$  is u.a.s but not e.s. for nominal system.

In particular, stability of the origin is not robust to smooth perturbations with arbitrary small linear growth bound.

Example. Scalar system,  $\gamma > 0$   $\dot{x} = -x^3 + \gamma x$

$x=0$  is g.a.s for  $\dot{x} = -x^3$  but not e.s.

$\|g(x)\| \leq \gamma \|x\|$ . However, linearization shows that  $x=0$  is unstable for perturbed system.  $-3x^2 + \gamma \Big|_{x=0} = \gamma > 0$ .

It is robust for a class of systems that we study next.

Example. Interconnected Systems.

$$\dot{x}_i = f_i(t, x_i) + g_i(t, x) \quad i=1, \dots, m,$$

$$x_i \in \mathbb{R}^{n_i}, n_1 + \dots + n_m = n, x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^n$$

Suppose  $f_i(t, 0) = g_i(t, 0) = 0 \quad \forall i$ .

unperturbed system is  $\dot{x}_i = f_i(t, x_i)$ .

Suppose origin is u.a.s. for each isolated subsystem.

What is stability of interconnected system?

Consider "composite Lyapunov function":

$$V(t, x) = \sum_{i=1}^m d_i V_i(t, x_i), \quad d_i > 0$$

suppose that for  $i=1, \dots, m$

$$\text{u.a.s.} \Rightarrow \left\{ \begin{array}{l} \frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t, x_i) \leq -d_i \phi_i^2(x_i) \\ \|\frac{\partial V_i}{\partial x_i}\| \leq B_i \phi_i(x_i) \end{array} \right.$$

$$\text{assumption: } \|g_i(t, x)\| \leq \sum_{j=1}^m \gamma_{ij} \phi_j(x_j)$$

for all  $t \geq 0$ ,  $\|x\| < r$ , for some  $d_i, B_i, \gamma_{ij} > 0$  where  $\phi_i: \mathbb{R}^{n_i} \rightarrow \mathbb{R}$  are positive definite and continuous.

$$\dot{V}(t, x) \leq -\frac{1}{2} \phi^T (D S + S^T D) \phi$$

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_m \end{pmatrix}, \quad D = \text{diag}(d_1, \dots, d_m)$$

$S$  is an  $m \times m$  matrix defined by

$$(S_{ij}) = \begin{cases} \alpha_i - B_i \gamma_{ii} & i=j \\ -B_i \gamma_{ij} & i \neq j \end{cases} \quad \begin{matrix} \text{measures degree of stability} \\ \text{of isolated subsystem} \end{matrix}$$

$\rightarrow$  represent the strength of interconnection

$\rightarrow S$  has nonpositive off diagonal elements.

Want positive diagonal matrix  $D$  so that  $Ds + s^T D > 0$ .

This is true if  $S$  is an M-matrix, which means that all the leading principal minors are positive. (Lemma 9.7).

(Exe 9.22) Diagonally dominant matrices with nonpositive offdiagonal elements are M-matrices.

This is sufficient condition for u.a.s. of origin of interconnected system (Thm 9.2).

Non-Vanishing Perturbation ( $g(t,0) \neq 0$ )

$g(t,0) \neq 0$ , so  $x=0$  is not an eq.pt. of perturbed system. In case of e.s (lemma 9.2) and u.a.s (lemma 9.3) of origin for nominal system, get uniform ultimate boundedness of solutions of perturbed system.

This is a robustness property of nominal system having e.s. eq.pt at the origin.

Lemma 9.2.  $x=0$  : e.s for  $\dot{x} = f(t,x)$ . Let  $V(t,x)$  be a Lyapunov function that satisfies:

$$c_1 \|x\|^2 \leq V(t,x) \leq c_2 \|x\|^2 \quad (1)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3 \|x\|^2 \quad (2)$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\| \quad (3)$$

In  $[0,\infty) \times D$  where  $D = \{x \in \mathbb{R}^n : \|x\| < r\}$ . Suppose the perturbation term  $g(t,x)$  satisfies

$$\|g(t,x)\| \leq s < \frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \text{ or}$$

$\forall t > 0$ ,  $\forall x \in D$  & some positive constant  $\theta < 1$ . Then, for all  $\|x(t_0)\| < \sqrt{\frac{c_1}{c_2}} r$ , the solution  $x(t)$  of the perturbed system satisfies

$$\|x(t)\| \leq K e^{-\delta(t-t_0)} \|x(t_0)\| \quad \forall t_0 \leq t < t_0 + T$$

$$\|x(t)\| \leq b \quad \forall t \geq T+t_0$$

for some finite  $T$ , where

$$K = \sqrt{\frac{c_2}{c_1}}, \quad \delta = \frac{(1-\theta)c_3}{2c_2}, \quad b = \frac{c_4}{c_3} \sqrt{\frac{c_2}{c_1}} - \frac{\theta}{\delta}$$

① Note that  $b$  is proportional to  $\theta$ : smaller perturbation  $\Rightarrow$  less s.s deviations from the origin.

②  $\frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \Theta r \rightarrow \infty$ , as  $r \rightarrow \infty$ . Therefore, if the assumptions hold globally, for all uniformly bounded disturbances, the solution of the perturbed system will be uniformly bounded. For any large  $\delta$ , we choose a large  $r$ .

Next lemma is a general form of this lemma. We'll prove next lemma.

**Lemma 9.3.**  $x=0$  : u.a.s for  $\dot{x}=f(t,x)$ . Let  $V(t,x)$  be a Lyapunov function that satisfies: (Thm 4.16 guarantees the existence of  $V$ )

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) \quad (1)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(\|x\|) \quad (2) \quad \alpha_i \in K$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq \alpha_4(\|x\|) \quad (3)$$

in  $[0,\infty) \times D$  where  $D = \{x \in \mathbb{R}^n : \|x\| < r\}$ . Suppose the perturbation term  $g(t,x)$  satisfies

$$\|g(t,x)\| \leq \delta < \frac{\theta \alpha_3 \alpha_2^{-1} \alpha_1(r)}{\alpha_4(r)}$$

$\forall t_0, \forall x \in D$  & some positive constant  $\theta < 1$ . Then, for all  $\|x(t_0)\| < \alpha_2^{-1} \alpha_1(r)$ , the solution  $x(t)$  of the perturbed system satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0) \quad \forall t_0 \leq t < t_0 + T, \quad \beta \in K$$

$$\|x(t)\| \leq \rho(\delta) \quad \forall t \geq T+t_0 \quad \rho \in K$$

for some finite  $T$ .  $\rho \in K$  is defined by  $\rho(\delta) = \alpha_1^{-1} \alpha_2 \alpha_3^{-1} \left( \frac{\delta \alpha_4(r)}{\theta} \right)$ .

① Note: We cannot say  $\frac{\theta \alpha_3 \alpha_2^{-1} \alpha_1(r)}{\alpha_4(r)} \xrightarrow[r \rightarrow \infty]{} \infty$  without having further information about  $K$  functions.

$\Rightarrow$  We cannot conclude that the solutions are bounded for any uniformly bounded perturbations.

*Proof.* Use  $V(t, x)$  as a Lyapunov candidate for the perturbed system. The derivative of  $V$  along the trajectories of the perturbed system satisfies:

$$\begin{aligned}\dot{V}(t, x) &\leq -\alpha_3(\|x\|) + \left\| \frac{\partial V}{\partial x} \right\| \|g(t, x)\| \\ &\leq -\alpha_3(\|x\|) + 8\alpha_4(\|x\|) \\ &\leq -(1-\theta)\alpha_3(\|x\|) - \theta\alpha_3(\|x\|) + 8\alpha_4(\|x\|) \\ &\leq -(1-\theta)\alpha_3(\|x\|)\end{aligned}$$

for any  $\|x\| \geq \alpha_3^{-1} \left( \frac{8\alpha_4(r)}{\theta} \right)$

Apply Thm 4.18.

To prove lemma 9.2., we only need to show that

$$\beta(\|x(t_0)\|, t-t_0) = K e^{-\gamma(t-t_0)} \|x(t_0)\|$$

(Do ex 4.51)